What Logical Knowledge is Needed to Account for Our Mathematical Knowledge

Sharon Berry

1. Introduction

Philosophers such as Sharon Street (2006) have pressed moral realists with an intuitive evolutionary debunking argument (EDA) along (very roughly) the following lines:

> *Moral EDA*
>
> If moral realism is true, it would be a massive mysterious coincidence if the forces of evolution, history, etc. gave human beings dispositions to form true beliefs about moral facts. In contrast, adopting more deflationary views like Humean sentimentalism or Street's constructivism lets one satisfyingly explain this accuracy without positing any such coincidence. Thus, we have *ceteris paribus* reason to reject moral realism in favor of some such more deflationary metaethical view (or give up pretensions to having moral knowledge).

A common strategy for replying to this argument (and similar worries) compares the special knowledge the moral realist takes us to have to mathematical knowledge. It is widely accepted, including by moral anti-realists, that we have significant mathematical knowledge.[1] However, as has been noted by philosophers from Plato to Benacerraf (1973) and Field (1980) and beyond, human accuracy about mathematics can seem quite mysterious. For example, we can't see or touch or taste mathematical objects or otherwise causally interact with them. Nor was there, presumably, evolutionary selection for correctly describing which abstract mathematical objects exist. And, more generally, we can seem to lack any kind of relationship to mathematical reality that could satisfactorily explain why our beliefs about mathematics would line up with mathematical facts.

Thus, the moral realist can make the following defensive suggestion:

> *Companions in Innocence Defense*
>
> The apparent extra coincidences which accepting moral realism forces us to posit are no worse than (and apparently similar in nature to) the coincidences that would be required for us to have got the kind of mathematical knowledge which we clearly have.

If the above Companions in Innocence claim is correct, this might suggest that the intuitions driving both moral and mathematical EDAs are generally untrustworthy or that there's some (undiscovered) solution to mathematical EDAs which can also be used to address EDAs against moral realism. Thus, the question of whether and how mathematical EDAs can be answered has wider philosophical importance.

[1] Or some form of accuracy that's very close to mathematical knowledge, such as fictionalism (see Field 1980 and Yablo 2005).

It’s a common thought that mathematicians are (for one reason or another) free to make any logical coherent pure mathematical posits they like, so we can reduce mathematical access worries to access worries about a certain kind of knowledge of logical coherence. But how helpful is this move? In this chapter, I’ll try to help readers clarify their views on this question in two ways. First, I’ll review and develop a coincidence avoidance framework for evaluating access worries and proposed solutions to them. Second, I’ll taxonomize and explain how different intuitive ideas about mathematics lead to different popular positions on what kind of knowledge of logical coherence would be needed to explain our mathematical knowledge in the above sense.

2. Formulating EDAs

2.1 Traditional Approaches and Easy Answers

Prima facie, it might seem that EDAs have something like the following form:

> *Explanatory Demand EDA*
>
> The realist’s inability to explain the (reliable) correlation between our judgments about some domain D and relevant facts about D casts doubt on their position.

However, (if taken literally) this way of formulating EDAs faces a problem. The problem is that a realist can answer the above EDA by providing *any* explanation (from premises the realist accepts) for human accuracy about the domain in question—even deeply intuitively unsatisfying ‘buck passing’ explanations that explain away apparent commitment to one mysterious coincidence by appealing to another!

For example, imagine that the moral realist turned out to be able to show the following. As a matter of physics, chemistry, and game theory, etc. (on planets involving earth-like basic chemistry), intelligent life could only evolve under certain conditions involving resource scarcity, payoffs for collaboration, etc.

Furthermore, any intelligent creatures evolving under those circumstances would face intense selective pressure to have moral sensibilities of a certain kind: to cooperate under certain circumstances and exercise a certain degree of nepotism, etc. And (it follows from first-order moral principles the realist believes that) these evolutionarily favored sensibilities also happen to be largely correct.[2] So, it follows that evolution occurring on planets like ours is reliably disposed to produce creatures with accurate moral sensibilities whenever it produces intelligent creatures at all.

In a sense, this theory would allow a moral realist to explain the correlation between human moral sentiments and relevant moral facts. However, this theory would do nothing to answer intuitive access worries. For it explains away one seeming coincidence (our being reliably disposed to form true moral beliefs) by appeal to another (that the game theoretically optimal moral sentiments for a certain environment turn out to be the ones that correctly reflect moral realist facts).

Various ways of reformulating or clarifying EDAs to avoid this problem have been tried (and run into trouble).

---

[2] More pedantically, they are sufficiently accurate to provide a reliable basis for forming true beliefs about moral realist facts (when suitably reflected on, etc.).

For example, at first glance one might be tempted to say the above explanations don't answer EDAs because they beg the question by appealing to assumptions (e.g., our acceptance of true mathematical axioms) which anyone pressing an access worry about moral realism/mathematical knowledge would deny. But this seems to transform EDAs into a general skeptical demand which most philosophers are already committed to rejecting. EDAs are interesting because they seem to provide a more powerful and interesting challenge than general skepticism because they seem to reveal an internal tension in the realist's views, not just that (like everyone else who has beliefs) the realist accepts some claims which can't be justified or explained from indubitable premises.

Alternately, one might reformulate EDAs as claims that the realist is committed to violating some kind of general epistemic constraint, requiring causal or explanatory connection between justified/knowledgeable beliefs and the subject matter of these beliefs, like the following:

- If S knows that P, then S's belief that P is caused by the fact that P. (Benacerraf 1973)
- If S rationally believes that P, then S must be open to the possibility that her belief causes or is caused by (or grounds or is grounded in) P. (Korman & Locke 2020)

However, finding an otherwise plausible such constraint has proved difficult. Note that, for example, the above constraints (famously) seem to rule out knowledge of the future: my belief that the sun will rise tomorrow isn't caused by or grounded in the fact that the sun will rise tomorrow (or *vice versa*). These constraints also threaten our knowledge of claims like[3] ‘A number is even if and only it if it’s divisible by 2’. For the fact that even numbers are divisible by two isn’t caused or constituted by my beliefs or *vice versa*.[4] We can often intuitively address EDA worries by giving a metasemantic explanation for human accuracy, rather than pointing out any kind of causal or grounding relationship. For example, one might say that our knowledge that ‘even numbers have the form $2k$ and odd numbers the form $2k + 1$’ doesn’t seem to require benefiting from a mysterious coincidence (and hence doesn’t raise an EDA worry) because if we’d been inclined to talk the other way, we’d have meant different things by ‘even’ and ‘odd’ so that we still expressed a truth. So, we probably don’t want to formulate EDAs in a way that rules such explanation out.

2.2 The Benefits of Being Lazy

Instead, I propose that we should be (strategically) lazy and formulate EDAs by appeal to *informal coincidence avoidance intuitions*. Such informal coincidence recognition abilities are already widely accepted and fruitfully used in science. In matters of *a priori* theory choice, we take ourselves to have *ceteris paribus* reason to favor theories that are committed to fewer coincidences (without further conceptual analysis of what it takes to there to be a coincidence/for some regularity to cry out for further explanation). And we can invoke such intuitions to formulate an EDA along the following lines:

Coincidence Avoidance Formulation of EDA

[3] Note that the truth of this claim doesn’t require that there are any numbers.

[4] Cf. Boghossian (1996) on metaphysical vs. epistemic analyticity.

> A realist theory faces an EDA to the extent that combining this theory with uncontroversial background beliefs (including claims about evolutionary history) forces us to posit *some* significant extra coincidence (involved in the match between human beliefs and facts about the domain in question) which could be avoided by adopting a less realist view.

Thinking about EDAs in this way nicely explains why the trivial explanation for human moral accuracy imagined above wouldn't suffice to answer EDAs. The problem is that this trivial explanation dispels one apparent coincidence (humans have largely correct moral sentiments) by appealing to another (the evolutionarily optimal degree moral sentiments in environments allowing for the development of intelligent life on earthlike planets turn out to be the ones that get moral realist facts right), which is left unexplained. So, this explanation doesn't change the appearance that adopting moral realism forces one to posit *some* significant extra coincidence.[5] The problem with the above easy explanations isn't that they appeal to some controversial belief the realist holds and the access worrier rejects, it's that they appeal to additional controversial coincidences the realist accepts.

Note that, blocking trivial responses to EDAs in this way doesn't collapse EDAs into a mere skeptical demand. For EDAs understood in terms of coincidence avoidance still point us to an *internal tension* in the realist's view. Someone pressing an EDA appeals to intuitions about coincidence reduction in theory choice which the realist shares. They suggest that the realist is—

[5] I.e., some coincidence that could be avoided by adopting some comparably attractive less realist alternatives to this view.

by their own lights—committed to positing extra coincidences that could be avoided by adopting certain (comparably attractive) more deflationary alternative views.

I suspect we can give a related diagnosis of many of the intuitively unsatisfying answers to EDAs in the recent literature on 'third factor explanations of human accuracy about moral facts.[65]

---

[6] A very simple example of such an unsatisfying explanation is Linnebo's (2006) "Boring explanation" of human mathematical accuracy, which (basically) explains mathematicians' tendency to have true beliefs by appeal to the idea that they reliably believe things that can be derived from the ZFC axioms, which are all true, together with the fact that all logically necessary consequences of truths are truths. This story intuitively fails to answer access worries because it explains away one apparent coincidence (disposition to reliably accept true claims as mathematical theorem) by appeal to another (acceptance of entirely true mathematical axioms).

For a more interesting case of such a third-factor explanation, consider David Enoch's (2010) intuitively unsatisfying explanation of human moral accuracy summarized below:

Enoch's Third Factor

> We evolved to have moral sentiments that promote survival. Many (but not all) things that promote survival happen to be (in the robust metaethical realist sense) morally good. The fact that survival is good explains the correlation between our normative beliefs and the normative facts.

I think Enoch's third-factor answer to moral EDAs faces a dilemma. Either it fails to account for the degree of moral accuracy moral realists take us to have or (if supplemented in a natural way) it fails like Linnebo's Boring Explanation because it explains one apparent coincidence by appeal to another, and thus doesn't reduce the total number of coincidences the realist is apparently committed to positing.

First, suppose we read Enoch to simply be saying that many (but by no means all) things that are morally good also promote survival. For example, looking out for children is both morally good and survival promoting, and one might list many other such things. This simple story can explain how many but not all of our moral beliefs are correct, and (unlike Linnebo's Boring Explanation) it plausibly has the merit of not appealing to any massive striking coincidence: many survival promoting things being good isn't more mysterious that many black things being shiny. So, it doesn't threaten to explain one big seeming coincidence by appeal to another.

However, this simple version of Enoch's story doesn't suffice to explain the accuracy about morals any normal moral realist is committed to. It's true that we only expect our moral beliefs to be largely true, not massively or exceptionlessly correct at the current moment—and perhaps Enoch's story explains this kind of accuracy. But we also take our accuracy about morals to go beyond this, and include things which Enoch's simple story does not explain. For example, one might think that we're disposed to be massively or perfectly correct about morals if given sufficient time for reflection, sympathizing, learning descriptive facts and approaching reflective equilibrium. And many realists think their tendency to have true moral beliefs isn't restricted to approving of actions that happen to be both morally good and survival promoting; they think they are decently accurate (or at least significantly better than chance) at reasoning about which of the things that don't promote survival are nonetheless morally good.

We might fix this problem by adding significant elaboration to Enoch's proposal by finding a number of additional "third factors" corresponding to all the things that aren't survival promoting which we value, until we get a theory which implies the disposition near-perfect match between moral judgment and moral fact under reflective equilibrium which many people take themselves to have. But once we have supplemented Enoch's theory in this way, it does seem that we're left with a massive apparent coincidence (for we are saying that many quite independent heterogenous things which are causally likely to be valued by intelligent creatures evolving in the environment we came from also happen to be good).

One might fear that taking this lazy approach to EDAs (not attempting to analyze what it takes to be committed to a coincidence or provide an explanation that banishes coincidence further) prevents the realist from plausibly defending themselves or makes it unlikely that access worries could be resolved. However, as I argue in Berry (2020), I think this is not the case.

For example, thinking about EDAs in accordance with the coincidence avoidance framework above also allows for a natural strategy for answering them. If EDAs are fundamentally arguments by appeal to impossibility intuitions (specifically intuitions that no explanation of human knowledge capable of dispelling apparent realist commitment to extra coincidences[7] is conceivable), then they can be answered by providing a kind of toy model explanation which (unlike the ones considered above) doesn't leave the realist intuitively committed to positing some extra coincidence.

Note that, on the story above, EDAs arise from a 'how possibly' question. We can't see how, e.g., mathematicians could possibly have acquired the accuracy they seem to have, without

In Berry (2020), I try to bring out a related implausibility intuition by considering the following Enoch-inspired third-factor explanation of (an aspect) human moral accuracy:

EV-MOR

> It is a robust fact that, in all circumstances conducive to the evolution of intelligence, natural selection favors the trait of advocating and valuing as being twice as generous with immediate family as with other individuals. Furthermore, it is morally correct to be (exactly) twice as generous with family, and this is a necessary truth.

[7] By this I mean coincidences that one could avoid by adopting a comparably attractive less realist view of our knowledge regarding the domain in question.

benefiting from some kind of striking coincidence that cries out for explanation. Yet adequate explanation[8] seems inconceivable. Accordingly, a natural way to answer to access worries would be to dissolve this feeling of inexplicability by providing a toy model [11, 30, 5], i.e., a sample explanation of how mathematical knowledge could have arisen. This sample explanation doesn't have to fit all known facts about how human mathematical knowledge actually arose. However, it does have to keep the key features of our actual situation that make adequate explanation seem inconceivable (e.g., our lack of causal contact with mathematical objects or logically possible worlds)[9]. It also cannot be buck-passing, in the sense that it explains one mysterious extra correlation the mathematical realist is committed to by appealing to another (e.g., one can't solve access worries merely by explaining mathematicians' acceptance of largely true theorems merely by appeal to their acceptance of largely true axioms).

2.3 Special Reasons for Distrust?

Perhaps some remarks of Clarke-Doane's (2016) suggest an objection to the lazy approach to EDAs advocated above. An objector might allow that we have a generally reliable faculty of spotting unattractive coincidences, and that it reliably guides our theory choice in the sciences.

[8] By 'adequate explanation' here, I mean explanation that banishes the appearance that accepting mathematical knowledge commits one to some 'extra' coincidence beyond those (if any) required to account for our possession of the three general purpose non-mathematical faculties noted as being presumed above.

[9] See Nozick (1981) and Cassam (2007) on blocking conditions, and Berry (2018b: pp 2288 n. 3).

However, they might argue that we have special reason to be suspicious of the coincidence avoidance intuitions that drive mathematical/moral EDAs, as follows.

In scientific cases, intuitively coincidence-banishing explanations tend to involve deducing some kind of modal stability from counterfactual supporting laws. Thus, one might think, a realist will succeed in giving a coincidence banishing explanation if they can deduce the modal stability of all striking correlations endorsed by their view from general-counterfactual supporting laws (which the realist accepts). And the necessity of mathematical/moral truths makes it easy for a realist defending against EDAs to derive the fact that human accuracy about mathematics/morality is modally stable in various ways (as below) from general laws:

- All of the closest possible worlds where mathematical facts are different (there are none) are ones where we have correspondingly different beliefs.
- Any story about how we're reliably inclined to have certain math beliefs would (when combined with relevant math facts and the fact that those are necessary truths) explain why we couldn't easily have formed false math beliefs.

So, one might think, human accuracy about math/morals can't genuinely cry out for further explanation, and the coincidence avoidance intuitions which EDAs draw upon must involve some kind of illusion.

However, I claim, this objection fails because there's actually strong independent reason to reject the key principle suggested above: that no coincidence can cry out for explanation, once the modal stability of all relevant regularities has been derived by appeal to general attractive laws.

Specifically, considering mathematical practices shows that certain regularities involving pairs of mathematical facts (both known to be necessary truths) can still cry out for explanation. Mathematical regularities (e.g., that some particular 7+ digit number turn out to play a special role in two apparently unrelated areas of mathematics[10]) can intuitively cry out for explanation. And mathematicians seem to rationally and fruitfully respond to this cry (favoring the hypothesis that some theorem more closely connecting the two facts and explaining the regularity exists, and guiding their research accordingly).

In these cases where regularities within pure mathematics cry out for explanation, both sides of the regularity have already been derived from general mathematical principles (like the ZFC axioms) that are necessary truths. So we already have perfect modal/counterfactual stability of the kind invoked above. Thus, we already have strong reason to think that coincidental seeming regularities can cry out for further explanation, even when their modal stability has been perfectly explained. And one might further suspect that, because of the connection between EDAs and coincidence avoidance intuitions, we shouldn't hope to cash out EDAs in terms of demands to demonstrate any kind of modal stability condition.

[10] Recent Fields-medal winning research was inspired by noting a regularity of this kind and looking for a theorem to explain it. See Klarreich (2017).

3. Mathematical EDAs and Knowledge of Logical Coherence

3.1 The Structuralist Consensus

With this coincidence avoidance approach in mind, let's now turn to mathematical EDAs. In this section, I'll discuss how a certain feature of contemporary mathematical practice suggests that (for one reason or another) mere knowledge of logical coherence would suffice to give us reliable true beliefs about mathematics.

This approach appeals to the following common idea about mathematicians' freedom, suggested by mathematical practice itself. Contemporary mathematical practice seems to allow mathematicians freedom to accept almost any logically coherent total collection of pure mathematical axioms they like.[11]

Mathematician-turned-philosopher Julian Cole puts the point as follows:

> Reflecting on my experiences as a research mathematician [some] things stand out. First, the frequency and intellectual ease with which I endorsed existential pure mathematical statements and

[11] Here I omit various details about spelling this out. For example, for these purposes, candidate 'pure mathematical axioms' are understood to be implicitly content restricted to the new mathematical structure in question (e.g., mathematicians who accept the Peano axioms characterizing the natural numbers don't accept that literally everything has a successor, but only that every number has a successor). So, endorsing mathematicians' freedom in the sense above does not commit one to the far more controversial claim that mathematicians would reliably form true beliefs if they adopted axioms implying sentences like "$\forall x \forall y x = y$". See Berry (2018b, 2022) for a little more detail on this point.

> referred to mathematical entities. Second, the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. (Cole 2009 : 589)

A related feature of contemporary mathematical practice which (also) contrasts with traditional moral realist understandings of moral talk concerns interpretations of disagreement. Traditional moral realists *aren't* inclined to say that, e.g., people with different logically coherent ways of using a term with the action-guiding role of 'permissible' are just getting at a different aspect of moral reality.[12] In contrast, (contemporary) mathematicians *are* inclined to say those who employ different logically coherent axioms have equally true beliefs about a different part of mathematical reality.

Unsurprisingly, philosophers of many different stripes have been inclined to follow mathematicians' lead and thus vindicate (in one way or another) the points about mathematicians' freedom above. Thus, they have accepted the following point, which I'll call the structuralist consensus:

> *Structuralist Consensus*
>
> Mathematicians can introduce any (or almost any) logically coherent stipulations defining a (pure) mathematical structure they wish.

[12] Cf. the Moral Twin Earth literature: e.g., Horgan & Timmons (1991).

Different philosophers have developed views which support the structuralist consensus in different ways. For example, here are a few such views in the contemporary philosophy of mathematics literature:

- Modal Structuralism: mathematical claims really express modal claims like 'it's logically possible for there to be objects satisfying certain axioms and necessarily, if there were objects satisfying such and such axioms then …'. That is, the true logical structure of mathematical claims (or their best Carnapian explication) is something $\Diamond D \wedge \Box(D \rightarrow \Phi)$, where the $\Diamond$ expresses logical possibility and the $\Box$ expresses logical necessity.
- Plentiudinous Platonism: the mathematical universe is very large, as per Balaguer (2001) and perhaps Neo-Fregean views[13] and classic set theoretic foundationalism on which all structures of interest can be identified with certain sets within a large hierarchy of sets as per Bourbaki.
- Weak Quantifier Variance: we have some freedom to choose how our language will 'carve up' the world into objects, including starting to talk in terms of additional objects (Hirsch 2011; Thomasson 2015; Berry 2015, forthcoming b).
  - e.g., a stipulation introducing complex numbers might attempt to secure the truth of some sentence S that conjoins the claim that every pair of real numbers r_1, r_2 corresponds to a complex number $r_1 + ir_2$ (with $r_1 + 0i = r_1$) with the rules for complex multiplication and

[13] These tend to also require conceptions of mathematical structures to take a specific form, that of abstraction principles.

addition.[14] This stipulation might also fix that the complex numbers are not to be identical to any physical objects, people, etc.

If some version of the structuralist consensus is right, then plausibly you can reduce EDAs for math to EDAs about logical coherence knowledge.

However, the task of accounting for our possession of the latter knowledge (without positing any mysterious coincidence that deniers of mathematical knowledge could avoid) is not trivial. Note that we need knowledge of which axioms describing pure mathematical structures are logically coherent ($\Diamond\Phi$ knowledge). And such knowledge can't be gotten by mere first-order logical deduction. For example, FOL deduction won't tell you that anything is logically coherent, e.g., $\Diamond(\exists x)(\exists y)(\neg x = y)$.

Recent works like Berry (2018b) have argued that *some* knowledge of logical coherence could be coincidence-banishingly explained by appeal to a kind of abductive mechanism, if working faculties of sensor perception, abduction/inference to the best explanation and first-order logical deduction (i.e., some very basic logical knowledge that poses less of an intuitive access worry) could be taken for granted. Note that the laws of logical possibility are supposed to be subject matter neutral, constraining the behavior of all objects and relations—from numbers to apples to ghosts or genres of novels. So, there's some hope that we could (in effect) abductively learn[15] general laws of logical possibility from dealing with non-mathematical objects, and then

[14] Here I take these addition and multiplication rules to be written in a way that implies that each complex numbers without an imaginary part is identical to the corresponding real number, as expected.

[15] And something analogous could happen at the level meme selection or gene selection.

apply them to deduce possibility claims about the very large and complex structures studied in (actualist or potentialist) mathematics—much as one could learn laws of physical possibility from experiments on the earth with pendulums, etc. and then apply them in space.

However, an important question arises about how *much* knowledge of logical possibility is needed to account for the kind of mathematical knowledge we seem to have. How much logical coherence knowledge would the above abductive story need to be able to deliver in order to settle mathematical access worries?

Opinions vary on this topic, so we'll see that there are two interesting dimensions of variation within the structuralist consensus:

- Mathematical metaphysics and ontology: are there mathematical objects? (The Plenitudinous Platonist and Quantifier Variantist will say "yes," and the modal structuralist will say "no.") Why are mathematicians free to introduce arbitrary logically coherent pure mathematical posits?
- Truth value and realism: how rich is our conception of mathematical structures like the natural numbers, hence how much knowledge of logical coherence (and perhaps reference to non-first-order logical notions) is needed to explain our mathematical knowledge?[16]

[16] A third dimension of variation one might list concerns how many mathematical claims one takes us to have knowledge of as opposed to, e.g., merely conditional knowledge that certain claims are derivable from

I won't argue for a position on either of the above questions here. Instead, I'll review and discuss the motivations for some different common positions regarding the latter bundle of questions: what our conceptions of mathematical structures are like—and hence what kind of $\lozenge\Phi$ and $(\Phi \rightarrow \Psi)$ knowledge is needed to answer mathematical EDAs.

3.2 How Much Logical Knowledge?

So, let's now turn to a menu of options regarding what our conceptions of mathematical structures are like, e.g., what primitive logical resources may be employed in stating them. I will present these options in order, so that we get a kind of ladder of increasing costs and benefits along certain dimensions. Specifically, we will see that increasing the logical (and other) resources one takes to be usable in stating our conceptions of mathematical structures like the natural numbers, has the benefit of allowing one to make sense of more truth-value realism about mathematics. But, on the other hand, it has the cost of raising concerns about reference to logical vocabulary and increasing the amount of logical coherence knowledge needed to account for the mathematical knowledge we seem to have via the structuralist consensus.

3.2.1 Finite First-Order Logical Conceptions

First, you might say that all our conceptions of math objects like the natural numbers and sets can be articulated by a sentence using first-order logical vocabulary and mathematical vocabulary.

---

certain axioms (without any assumption that those axioms are logically coherent). But I won't dwell on this because (with some exceptions) there doesn't seem to be huge variation on this matter.

Thus, for example, you might take our conception of the natural numbers to be something like the conjunction of the finitely many sentences in Q (Robinson's Arithmetic).

On this view, accounting for the mathematical knowledge via the structuralist consensus will only require recognizing $\Diamond\varphi$ facts where φ is a sentence in the language of first-order logic, like $\Diamond Q$.

A weakness of this view is that we seem willing to accept all instances of the induction schema below. That is, we're inclined to accept all the infinitely many sentences got by substituting a formula φ (where φ can be any sentence in the language of first-order logic with a single free variable) into the schema below. And the widely used Peano Axioms (PA) for number theory include all instances of this schema where the relevant φ uses only number theoretic vocabulary:

FOL Induction Schema

$$(\varphi(0) \wedge (\forall n)(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow (\forall n)(\varphi(n))$$

3.2.2 Recursively Axiomatizable First-Order Conceptions

One can address this weakness by allowing our conceptions of mathematical structures to include infinite collections of first-order logical axioms, provided that they are *recursively axiomatizable* (i.e., that one could, in principle, program a computer to determine whether any given first-order logical sentence belongs to the axioms included in the theory). This allows one to say that our conception of the natural numbers includes all the Peano Axioms, including the infinitely many instances of the induction schema above. One might also say that, whether or not we can fully state such infinite theories, we can implicitly grasp and accept them via being disposed to accept each axiom included in them.

On this view, accounting for mathematical knowledge via the structuralist consensus can require recognizing (so to speak) $\lozenge T$ facts, where T is some infinite recursively axiomatizable theory as above. So, for example, accounting for our knowledge of the natural numbers will require accounting for knowledge of things like (so to speak[17]) $\lozenge PA$. Now I will discuss some costs and benefits which apply to both approaches above.

A benefit of saying that our conception of mathematical structures consists in some collection of first-order logical axioms is that it makes access worries less severe. For, it turns out, all first-order logical theories are *logically incoherent* (require something logically impossible) if and only if they are syntactically inconsistent (i.e., contradiction can be derived from them).[18] So, if mathematicians could perform infinite tasks in finite time, they could (in principle) recognize and reject all logically inconsistent theories just by going through all possible proofs whose premises belong to the relevant theory. This helps with access worries somewhat, though the help is limited by the fact that mathematicians obviously don't do go through this infinite process before accepting candidate mathematical axioms as logically coherent.

[17] Note that (prima facie) $\lozenge T$ isn't really a sentence (unless sentences are allowed to include infinite conjunctions). A potential cost of this view is that extra work is needed to reconcile it with modal structuralist views in the structuralist consensus. For if we translate mathematical sentences as saying something like $\square(D \rightarrow \varphi)\lozenge D$, it would seem that we need a single sentence to slip into the place of D.

[18] Whenever you can't derive a contradiction from a first-order logical theory, there's a set theoretic model which makes that theory true (see Gödel 1930).

However, a significant cost of taking our conceptions of mathematical structures to consist in finite or even infinite recursively axiomatizable collection of first-order logical statements is famously shown by Gödel's Incompleteness Theorem. For the latter theorem tells us that any consistent such theory (which implies all the Peano Axioms, intuitive truths about the natural numbers) will fail to determine a truth value for some number theoretic sentence φ, and indeed fail to logically necessitate either φ or $\neg\varphi$.[19]

Accordingly, one cost of both views now being discussed is that they *prima facie* conflict with intuitions that there must be definite right answers to all questions in the language of number theory. Many people feel a strong intuition that there's a definite right answer to 'Are there infinitely many twin primes (i.e., prime numbers separated by only one number)?' and all such questions about sentences statable using only number theoretic relations and quantifiers restricted to the natural numbers. But on the pair of views under consideration now, one might have to say that some such questions are indeterminate. For, whatever conception of the numbers along these lines we have, there will be different possible ways of assigning extensions to 'number' and 'successor' (while holding fixed the meaning of all first-order logical vocabulary) which both satisfy all the axioms in our conception but yield different truth-values for some possible sentence.

Indeed, the problem gets worse. For Gödel's theorem actually shows that each FOL theory of the kind mentioned above fails to determine an answer to some Con(T) sentence. These are sentences that only use mathematical vocabulary, but intuitively say that no number codes a proof of '0=1' from premises in a certain algorithmically described first-order logical theory T. Thus

[19] The second claim follows by the Completeness theorem for first-order logic (see Gödel 1930).

we're disposed to accept (and treat as an *a priori*, conceptually central, truth that constrains acceptable intuitive interpretations of our number talk) a biconditional of the following form:

- Con(T) iff 0=1 isn't provable from the axioms of T.

Thus, if you accept that there are determinate facts about provability (and hence determinate truth-values for all claims on the right-hand side of this biconditional), there's some pressure to accept that there are also determinate truth values for all Con(T) sentences.

Thus, even if you are happy to allow that *some* number theoretic questions are indeterminate, accepting definite facts about the syntactic consistency of theories creates pressure to say that all such Con(T) sentences have definite truth-values. Thus, intuitions about provability motivate saying that our conception of mathematical structures can transcend the claim that mathematical objects and relations (like 'natural number' and 'successor') apply so as to satisfy some finite (or recursively axiomatizable) collection of first-order logical sentences true.

3.2.3 True ℕ Views

The next cluster of views I want to mention avoids the problems above by saying that we (somehow) have the conceptual resources to refer to a unique intended natural number structure in some special way, but lack any logical notions with the power of full second-order quantification. One might cite Kronecker's famous aphorism that "God made the natural numbers; all else is the work of man" (Gray 2008: 18) in this connection.

This can seem a little awkward insofar as (you might think) we grasp the meaning of the natural number structure by something like accepting some simple first-order logical principles together with a second-order logical induction axiom.[20] But if we can grasp this, then why can't we grasp full second-order quantification in other contexts? If we can grasp the intended natural number structure in this way, why can't we use the same tool when stating other conceptions of mathematical structures? However, philosophers have explored different ways of thinking about what our conception of the natural number structure might be like.

For one thing, works like Field (1989) have rather tentatively suggested different ways of using reference to physical objects to pick out an intended natural number structure. For example, Field tentatively suggests that if time has a certain structure, then we could use reference to temporal points and distance relations to pin down the structure we take the 'natural numbers' to have. And in Berry (forthcoming a), I suggest some ways that definite reference to a notion of physical possibility might be enough to pin down unique reference for one's natural number talk.

A different approach to conceptions of mathematical structures which allows us to uniquely pick out an intended natural number structure but not make sense of arbitrary second-order quantification involves a notion of 'sufficiently clear mental pictures'. In some interesting philosophical remarks motivating more precise formal proposals, Feferman (2012) considers the formulability of various mental pictures of mathematical structures. He suggests that we can have a sufficiently clear conception of the intended natural number structure "represented by the tallies |,||,|||." And he writes that our conception of the continuum in terms of points on a line is clearer

[20] Note that combining this principle this with PA^{-} (i.e., the finitely many Peano axioms aside from instances of the induction schema) suffices to pin down a unique natural number structure, in the sense that all set theoretic models/interpretations of these theories which interpret second-order quantification standardly are isomorphic to each other

than the set-theoretic conception of it in terms of arbitrary subsets of the integers. He writes: "we have a much clearer conception of arbitrary sequences of points on the Hilbert (or Dedekind, or Cauchy-Cantor) line, or at least of bounded strictly monotone sequence, than we do of arbitrary subsets of the line. And … we have a clearer conception of what it means to be an arbitrary infinite path through the full binary tree than of what it means to be an arbitrary subset of N, but in neither case do we have a clear conception of the totality of such paths, resp. sets" (Feferman 2012).

But Feferman suggests that neither of these conceptions are sufficiently clear to pick out a unique intended natural number structure.[21] However, as Koellner (2016) points out, one might argue that this concept of having a sufficiently clear mental picture (and the philosophical motivations behind taking such a different attitude to different kinds of mental pictures of mathematical structures) is arguably itself not sufficiently clear.[22]

[21] Note that for these purposes having a clear conception doesn't just mean being able to have some mental picture of that structure, e.g., imagining the hierarchy of sets by mentally picturing a *V*-shaped expanding column. For Feferman suggests that there's a way in which this picture represents itself as being fully determinate yet fails to be so determinate. He writes: "There is no problem to put oneself in the mental frame of mind of 'this is what the cumulative hierarchy looks like', for which one can see that such and such propositions including the axioms of ZFC are (more or less) obviously true. I have taught set theory many times and have presented it in terms of this ideal-world picture with only the caveat that this is what things are supposed to be like in that world, rather than to assert that's the way the world actually is" (Feferman 2012).

[22] Note that Feferman makes various precise formal proposals that draw the line concerning where we can vs. can't expect determinate right answers to all mathematical questions. The question at issue concerns the

What kind of logical knowledge we need to make sense of our mathematical knowledge will naturally depend on the details of the specific proposal one adopts within this family of views. But, speaking abstractly, we will need to account for knowledge of $\Diamond\Phi$ claims where Φ employs whatever extra expressive resources are added to first-order logic. Feferman's talk of mental pictures suggests a version of the structuralist consensus where accuracy about mathematics could be explained by an ability to determine when a mental picture (under some 'method of projection') represented something logically coherent, and when it necessitates the truth of various mathematical axioms.

3.2.4 Second-Order Logical Conceptions

Next, one might say that our conceptions of mathematical structures can employ second-order quantification (or something of similar expressive power) as well the standard first-order logical connectives.[23] A benefit of this approach is that it lets us say that our conception of the natural numbers includes something like the following second-order induction axiom[24]:

$$(\forall X)[(X(0) \land (\forall n)(X(n) \to X(n+1))) \to (\forall n)(X(n))]$$

---

philosophical motivation for assigning that significance to the various precise mathematical distinctions Feferman highlights.

[23] Note that the idea here is that our conceptions of a mathematical structure can appeal to second-order quantification (or the like) while taking this notion to already have a precise meaning, which can be grasped prior to any choice to accept or consider any conceptions of mathematical structures employing it.

[24] Here I use 0 to abbreviate the corresponding statement in terms of Russell's definite description.

This single axiom implies all the specific instances of the induction schema mentioned above. The second-order induction axiom above, when combined with first-order axioms, suffices to completely pin down the natural number structure, in the sense that any two ways of assigning extensions to ‘number’ and ‘successor’ while preserving the meaning of all logical connectives will have the same structure.[25] Note that it employs a second-order quantifier $\forall X$ to talk about ‘all collections of’ or ‘all possible ways of choosing from’ the objects in the domain of first-order quantification.

Thus, on this view, we can write a finite sentence PA_2 (Second-Order Peano Arithmetic) which logically necessitates the truth or falsehood of each sentence φ in the language of number theory. Note that facts about statements using second-order quantification can have logical consequences[26] which outstrip what we’re able to derive from them. Thus, the fact that PA_2 logically necessitates each sentence of number theory or its negation doesn’t mean that we’ll always be able to determine the truth-value of such sentences.

As noted above, adopting this view about our conceptions of mathematical structures tends to increase the amount of logical coherence knowledge needed to account for our seeming mathematical knowledge (and thence potentially the difficulty of answering access worries). For it allows that accounting for the mathematical knowledge we seem to have can require recognizing

---

[25] That is, it will be possible for a function to pair the ‘numbers’ on one interpretation to the numbers in another in a way that preserves how ‘successor’ (and hence plus and times) applies.

[26] That is, φ can have ψ as a logical consequence (logically necessitate that ψ in a sense which implies that e.g., every set theoretic model of φ which interprets second order quantification appropriately also satisfies ψ, even if the various axioms and inference rules we accept don’t let us derive ψ from φ.

$\lozenge\varphi$ facts, where φ is a statement involving second-order quantification. For example, someone who favors this approach will say that recognizing the logical coherence of our conception of the numbers requires recognizing a fact like $\lozenge PA_2$ rather than merely something like $\lozenge Q$.

Additionally, this approach can raise worries about reference to whatever additional logical vocabulary (e.g., full second-order quantification) is being cleared for use in stating our conception of mathematical structures.[27]

A different potential weakness of this view (which may incline one to allow even more powerful resources to our conceptions of mathematical structures) concerns what to say about set theory. In response to Russell's paradox, set theorists embraced an iterative hierarchy conception of sets. On the iterative hierarchy conception, all sets can be thought of as existing within a hierarchy built up in layers (that satisfy the well ordering axioms). There's an empty set (the set that has no elements) at the bottom, and each layer containing sets corresponding to all ways of choosing sets generated below this layer.[28] It follows from this conception of (what I'll call) the *width* of the hierarchy of sets, that whenever the hierarchy of sets contains a set x, it also contains sets corresponding to all possible ways of choosing some elements from x.

[27] Perhaps one can address some of this worry by making a referential companions-in-innocence argument which holds that reference to conditional logical possibility is no more mysterious or problematic than reference to non-Humean facts about physical possibility.

[28] The iterative hierarchy of sets is sometimes formulated to include limit stages which simply collect layers from below, but don't form any new sets. However, this variation makes no philosophical or mathematical difference, and it is easy to translate between the two ways of imagining the iterative hierarchy of sets being divided into layers

Interestingly, axioms stateable in second-order logic suffice to pin down a unique intended natural number structure (and hence right answers to number-theoretic questions) as well as a conception of the width of the hierarchy of sets and hence right answers to all questions which only depend on it, including the famous continuum hypothesis,[29] a conjecture whose truth-value is famously not determined by the standard ZFC axioms of set theory. But no widely accepted axioms in second-order logic suffice to fix a unique intended height for the hierarchy of sets.

Instead, we find ourselves in the following curious situation. Our naive conception of absolute infinity (the height of the actualist hierarchy of sets) turns out to be incoherent, not just unanalyzable.

Specifically, a very common intuitive conception of the hierarchy of sets says that the hierarchy of sets goes ‘all the way up’—so no restrictive ideas of where it stops are needed to understand its behavior. However, if the sets really do go ‘all the way up’ in this sense, then it would seem that they should satisfy the following naive height principle:

> *Naive Height Principle*
>
> For any way some things are well-ordered by some relation R, there is an ordinal (and a layer of the hierarchy of sets) corresponding to it.

But the layers of the hierarchy of sets are themselves well ordered, and there is no ordinal corresponding to this well-ordering, i.e., there is no ordinal which has the same order-type as the

[29] Are there sets intermediate in size between the natural numbers and the real numbers?

class of all ordinals. Thus, it would seem, the naive height ordering principle above can't be correct. And once we reject this naive conception, there's no obvious fallback conception that *even appears* to specify a unique intended height for the hierarchy of sets.

Accordingly, philosophers considered a range of responses to this puzzle. First, of course, one can embrace the indeterminacy and, e.g., just say we mean some structure that satisfies second-order logical versions of the ZFC axioms for set theory. This is a popular response, as most people have a much stronger intuition that there are definite right answers to questions about number theory than about set theory.

Second, one could (in principle) propose a different description, in the language of second-order logic, of a unique intended height for the hierarchy of sets, to replace the naive conception referenced above. However, no such proposal has gained significant popularity. And natural candidates for such a description of the intended height of the hierarchy of sets (e.g., saying that the hierarchy of sets is, in effect, *the shortest possible* structure satisfying the ZFC_2 axioms—which turn out to imply width constraints on the iterative hierarchy of sets) tend to conflict with the generality mathematicians want set theory to have (Hellman 1994).

3.2.5 Beyond Second-Order Logical Conceptions

Allowing conceptions of mathematical structures to employ logical vocabulary (hence as antecedent meaningful) resources going beyond second-order quantification opens up to more responses to questions about our conception of the height of the hierarchy of sets above.

Most simply, one could (in principle) say that we can somehow directly refer to a primitive notion of absolute infinity, which picks out a unique intended height of the hierarchy of sets. That is, one could take this notion to be a logical primitive on par with first-order logical connectives and second-order quantification. However, this option has little intuitive appeal; after rejecting the incoherent notion of the hierarchy of sets going 'all the way up' discussed above, few people would say they even *seem* to grasp a unique intended height for the hierarchy of sets, in the way that many of us feel we do seem to grasp a favored notion of 'all possible ways of choosing' (and hence an intended notion of second-order quantification). And, to my knowledge, no one takes this route.

Alternately, one can adopt a potentialist understanding of set theory. Potentialists can accept that there are definite right answers to all questions in the language of set theory, while denying that we have a conception of a unique intended height of the hierarchy of sets. For they interpret set theory as an exploration of how it would be (in some sense) possible for standard-width initial segments of the hierarchy of sets to be extended. They wind up needing extra resources that go slightly beyond second-order quantification to express such extendability claims.[30]

---

[30] Putnam (1967) develops this idea by thinking about how it would be possible to have objects forming intended models of certain axioms for set theory but leaves the details of what modal notion he wants to invoke somewhat vague. Later work by Hellman (1994) and Berry (2018a) proposes different ways of cashing this idea out by using a notion of logical possibility which has been argued to be an independently attractive primitive. Hellman uses logical possibility, plural quantification and mereology (to simulate second order relation quantification). I use a generalization of the logical possibility operator.

Parsons (1977, 2005, 2007), Linnebo (2010, 2013, 2018) and Studd (2019) take the idea in a different direction. Rather than thinking about how it would be logically possible for there to be objects

In this way, we see that members of the structuralist consensus can hold a range of different theories about the nature of our conceptions of mathematical structures, and these views correspond to different views on what logical coherence knowledge one needs to account for mathematical knowledge and hence perhaps how hard mathematical access worries are to solve.

4. Conclusion

In this chapter, I have advocated a 'lazy' approach to formulating EDAs, which appeals to informal coincidence avoidance intuitions that are widely accepted as good guides to theory choice in the sciences. I have noted how mathematical practice has inspired a structuralist consensus, on which mathematical EDAs can plausibly be reduced to EDAs concerning logical coherence, and abduction-based stories have been proposed to account for our possession of some knowledge of logical coherence. Finally, I've reviewed and contrasted certain popular and/or natural ideas about what our conceptions of mathematical structures are like—how they imply different things about how much logical coherence knowledge is needed to account for the kind of mathematical knowledge we seem to have in answering EDAs.

---

satisfying set-theoretic axioms, Linnebo and Studd say that whatever sets exist (if any) exist necessarily. But they cash out set theory in terms of how it would be interpretationally possible for a hierarchy of sets to grow, where this involves something like successively reconceptualizing the world in terms of longer and longer actualist hierarchies of sets.

I personally hold a view on the extreme realist/generous end regarding what logical vocabulary our conceptions of mathematical structures can employ,[31] and I think that even on this generous understanding of our conception of mathematical structures, the kind of broadly abductive story invoked above can account for enough logical knowledge to answer mathematical EDAs. However, this degree of optimism about the power of abduction/IBE to reliably extend our beliefs about logical coherence is controversial. So, it's worth noting that philosophers who reject the kinds of truth-value realist intuitions that I've noted motivate this understanding of our conceptions of mathematical structures might need less optimism about the power of abduction to answer EDAs along the lines mentioned above.[32]

References

Balaguer, Mark. 2001. *Platonism and Anti-Platonism in Mathematics*. Oxford: Oxford University Press, 2001.

Benacerraf, Paul. 1973. "Mathematical Truth," *The Journal of Philosophy* 70: 661–680.

Berry, Sharon. 2015. "Chalmers, Quantifier Variance and Mathematicians' Freedom." In A. Torza (ed.), *Quantifiers, Quantifiers, and Quantifiers. Themes in Logic, Metaphysics and Language*, 191–219. Dordrecht: Springer.

Berry, Sharon. 2018a. "Modal Structuralism Simplified," *Canadian Journal of Philosophy* 48 (2): 200–222.

[31] I think that we can somehow refer to a conditional logical possibility operator, which can do all the work of second-order quantification and make the extendability claims needed for potentalist set theory.

[32] I am grateful to Mary Leng for helpful comments on an earlier version of this paper.

Berry, Sharon. 2018b. "(Probably) Not companions in Guilt," *Philosophical Studies* 175 (9): 2285–2308.

Berry, Sharon. 2020. "Coincidence Avoidance and Formulating the Access Problem," *Canadian Journal of Philosophy* 50(6): 687–701.

Berry, Sharon. 2022. *A Logical Foundation for Potentialist Set Theory*. Cambridge: Cambridge University Press.

Berry, Sharon. Forthcoming a. "Physical Possibility and Determinate Number Theory," *Philosophia Mathematica*.

Berry, Sharon. Forthcoming b. "Quantifier Variance, Mathematicians' Freedom and the Revenge of Quinean Indispensability Worries," *Erkenntnis*.

Boghossian, Paul. 1996. "Analyticity Reconsidered," *Noûs* 30 (3): 360–391.

Cassam, Quassim. 2007. *The Possibility of Knowledge*. Oxford: Oxford University Press.

Clarke-Doane, Justin. 2016. "What Is the Benacerraf Problem?" In F. Pataut (ed.), *New Perspectives on the Philosophy of Paul Benacerraf: Truth, Objects, Infinity*, 17–43. Dordrecht: Springer.

Cole, Julian. 2009. "Creativity, Freedom, and Authority: A New Perspective on the Metaphysics of Mathematics," *Australasian Journal of Philosophy* 87 (4): 598-608

Enoch, David. 2010. "The Epistemological Challenge to Metanormative Realism: How Best to Understand It, and How to Cope with It," *Philosophical Studies* 148 (3): 413–438.

Feferman, Solomon. 2012. "Is the Continuum Hypothesis a Definite Mathematical Problem?" Paul Bernays Lectures, ETH Zürich.

Field, Hartry. 1980. *Science Without Numbers: A Defense of Nominalism*. Princeton: Princeton University Press.

Field, Hartry. 1989. *Realism, Mathematics & Modality*. Oxford: Blackwell.

Gray, Jeremy. 2008. *Plato's Ghost: The Modernist Transformation of Mathematics*. Princeton: Princeton University Press.

Gödel, Kurt. 1930. "Die Vollst¨andigkeit der Axiome des logischen Funktionenkalküls," *Monatshefte für Mathematik und Physik* 37 (1):349–360, 1930.

Hellman, Geoffrey. 1994. *Mathematics Without Numbers*. Oxford: Oxford University Press.

Hirsch, Eli. 2011. *Quantifier Variance and Realism: Essays in Metaontology*. Oxford: Oxford University Press.

Horgan, Terence & Mark Timmons. 1991. "New Wave Moral Realism Meets Moral Twin Earth," *Journal of Philosophical Research* 16: 447–465.

Klarreich, Erica. 2017. "Mathematicians Chase Moonshine's Shadow." In M. Pitici (ed.), *The Best Writing on Mathematics 2016*. Princeton: Princeton University Press.

Koellner, Peter. 2106. "Infinity Up on Trial: Reply to Feferman," *The Journal of Philosophy*, 113 (5/6): 247–260.

Korman, Dustin & and Dustin Locke. 2020. "Against Minimalist Responses to Moral Debunking Arguments," *Oxford Studies in Metaethics* 15: 309–332.

Linnebo, Øystein. 2006. "Epistemological Challenges to Mathematical Platonism," *Philosophical Studies* 129 (3): 545–574.

Linnebo, Øystein. 2010. "Pluralities and Sets," *The Journal of Philosophy* 107 (3): 144–164.

Linnebo, Øystein. 2013. “The Potential Hierarchy of Sets,” *Review of Symbolic Logic* 6 (2): 205–228.

Linnebo, Øystein. 2018. *Thin Objects*. Oxford: Oxford University Press.

Nozick, Robert. 1981. *Philosophical Explanations*. Cambridge, MA: Harvard University Press.

Parsons, Charles. 1977. “What Is the Iterative Conception of Set?” In R. Butts & J. Hintikka (eds.), *Logic, Foundations of Mathematics, and Computability Theory: Part One of the Proceedings of the Fifth International Congress of Logic, Methodology and Philosophy of Science, London, Ontario, Canada-1975*, 335–367. Dordrecht: Reidel.

Parsons, Charles. 2005. *Mathematics in Philosophy*. Ithaca: Cornell University Press.

Parsons, Charles. 2007. *Mathematical Thought and Its Objects*. Cambridge: Cambridge University Press.

Putnam, Hilary. 1967. “Mathematics without Foundations,” *The Journal of Philosophy* 64 (1): 5–22.

Street, Sharon. 2006. “A Darwinian Dilemma for Realist Theories of Value,” *Philosophical Studies* 127 (1): 109–166.

Studd, J. P. 2019. *Everything, More or Less: A Defence of Generality Relativism*. Oxford: Oxford University Press.

Thomasson, Amie. 2015. *Ontology Made Easy*. New York: Oxford University Press.

Yablo, Stephen. 2005. “The Myth of Seven.” In M. Kalderon (ed.), *Fictionalism in Metaphysics*, 88–115. Oxford: Oxford University Press.